MATH 303 – Measures and Integration
Homework 3

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f : X \to [0, \infty]$ is a measurable function. Define $\nu : \mathcal{B} \to [0, \infty]$ by

$$\nu(E) = \int_E f \, d\mu.$$

Prove that ν is a measure.

Solution: The function $f \cdot \mathbb{1}_\emptyset$ is the zero function, which is simple, so we can integrate

$$\nu(\emptyset) = \int_X f \cdot \mathbb{1}_\emptyset \, d\mu = \int_X 0 \, d\mu = 0 \cdot \mu(X) = 0.$$

It remains to check that ν is countably additive. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of disjoint measurable sets. Let $E = \bigcup_{n\in\mathbb{N}} E_n$. Then $\mathbb{1}_E = \sum_{n\in\mathbb{N}} \mathbb{1}_{E_n}$. Therefore, by Theorem 3.12,

$$\nu(E) = \int_X f \cdot \mathbb{1}_E \, d\mu = \int_X \sum_{n\in\mathbb{N}} (f \cdot \mathbb{1}_{E_n}) \, d\mu = \sum_{n\in\mathbb{N}} \int_X f \cdot \mathbb{1}_{E_n} \, d\mu = \sum_{n\in\mathbb{N}} \nu(E_n).$$

Problem 2. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be an integrable function. Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $E \in \mathcal{B}$ and $\mu(E) < \delta$, then $\left|\int_E f \, d\mu\right| < \varepsilon$.

Solution: There are several methods for solving this problem. We give four different methods below: one using the hint provided in the homework, one using simple functions, one using bounded functions and monotone convergence, and one using the dominated convergence theorem. Some of the solutions use material that was not yet covered at the time of this assignment; these steps are marked with an asterisk.

Method 1: Using the hint and problem 1.

By the triangle inequality for integrals (Proposition 3.16), it suffices to prove the conclusion for $|f|$ in place of f. Then since $|f|$ is a nonnegative function, the set function $\nu : \mathcal{B} \to [0, \infty]$ defined by $\nu(E) = \int_E |f| \, d\mu$ is a measure by Problem 1. Moreover, $\nu(X) = \int_X |f| \, d\mu < \infty$, since f is integrable. Finally, if $E \in \mathcal{B}$ and $\mu(E) = 0$, then

$$\nu(E) = \int_X f \cdot \mathbb{1}_E \, d\mu \leq \int_X (\infty \cdot \mathbb{1}_E) \, d\mu = \infty \cdot \mu(E) = \infty \cdot 0 = 0. \tag{1}$$

Let $\varepsilon > 0$. Suppose for contradiction that there exists a sequence $(E_n)_{n\in\mathbb{N}}$ with $\mu(E_n) \to 0$ such that $\nu(E_n) \geq \varepsilon$ for every $n \in \mathbb{N}$. By refining to a subsequence, we may assume $\mu(E_n) < 2^{-n}$ for each $n \in \mathbb{N}$ so that $\sum_{n\in\mathbb{N}} \mu(E_n) < \infty$. Let $A_N = \bigcup_{n\geq N} E_n$. Then $A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing sequence, and $\mu(A_1) \leq \sum_{n\in\mathbb{N}} \mu(E_n) < \infty$, so by continuity from above (Proposition 2.15 in the lecture notes),

$$\mu\left(\bigcap_{N\in\mathbb{N}} A_N\right) = \lim_{N\to\infty} \mu(A_N) \leq \lim_{N\to\infty} \sum_{n\geq N} \mu(E_n) = 0.$$

As shown in (1), every set of μ measure zero is also a set of ν measure zero, so

$$\nu\left(\bigcap_{N\in\mathbb{N}} A_N\right) = 0. \tag{2}$$

But since $\nu(A_1) \leq \nu(X) < \infty$, using continuity from above and monotonicity of ν, we have

$$\nu\left(\bigcap_{N\in\mathbb{N}} A_N\right) = \lim_{N\to\infty} \nu(A_N) \geq \liminf_{N\to\infty} \nu(E_N) \geq \varepsilon. \tag{3}$$

The statements (2) and (3) are contradictory. Thus, there exists $\delta > 0$ such that if $E \in \mathcal{B}$ and $\mu(E) < \delta$, then

$$\left|\int_E f\ d\mu\right| \leq \nu(E) < \varepsilon.$$

Method 2: Simple functions.

Let $\varepsilon > 0$. By the definition of the integral,

$$\int_X |f|\ d\mu = \sup\left\{\int_X s\ d\mu : 0 \leq s \leq |f|, s \text{ simple}\right\}.$$

Since f is assumed to be integrable, we have $\int_X |f|\ d\mu < \infty$, so there exists a simple function $s : X \to [0,\infty)$, $0 \leq s \leq |f|$, such that

$$\int_X s\ d\mu > \int_X |f|\ d\mu - \frac{\varepsilon}{2}.$$

Simple functions take only finitely many different values, so s has a maximum value $M = \max_{x\in X} s(x)$. Let $\delta = \frac{\varepsilon}{2M}$. If $E \in \mathcal{B}$ and $\mu(E) < \delta$, then

$$\begin{aligned}
\left|\int_E f\ d\mu\right| &\leq \int_E |f|\ d\mu && \text{(triangle inequality for the integral)}\\
&= \int_E s\ d\mu + \int_E (|f| - s)\ d\mu && \text{(additivity of the integral*)}\\
&\leq \int_E M\ d\mu + \int_X (|f| - s)\ d\mu && \text{(monotonicity of the integral)}\\
&= M\mu(E) + \int_X |f|\ d\mu - \int_X s\ d\mu && \text{(additivity of the integral*)}\\
&< M\delta + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Method 3: Monotone convergence theorem.

Let $\varepsilon > 0$. For $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| \leq n; \\ n, & \text{if } |f(x)| > n. \end{cases}$$

That is, $f_n(x) = \min\{|f(x)|, n\}$. Then $0 \leq f_1 \leq f_2 \leq \ldots$ is an increasing sequence of measurable functions, and $f_n(x) \to |f(x)|$ for every $x \in X$. Therefore, by the monotone convergence theorem,

$$\int_X |f| \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Let $n \in \mathbb{N}$ such that $\int_X |f| \, d\mu < \int_X f_n \, d\mu + \frac{\varepsilon}{2}$. We then complete the argument the same way as in Method 2, using f_n in place of s and n in place of M.

Method 4: Borel–Cantelli and dominated convergence.

Let $\varepsilon > 0$. Suppose for contradiction that no such δ exists. Then there is a sequence of measurable sets $(E_n)_{n\in\mathbb{N}}$ such that $\mu(E_n) < 2^{-n}$ and $\left|\int_{E_n} f \, d\mu\right| \geq \varepsilon$. Let $f_n = f \cdot \mathbb{1}_{E_n}$, and let $B = \{x \in X : f_n(x) \not\to 0\}$. Then $B \subseteq \{x \in X : x \in E_n \text{ for infinitely many } n \in \mathbb{N}\}$, so $\mu(B) = 0$ by the Borel–Cantelli lemma. Hence, $f_n \to 0$ almost everywhere. Moreover, $|f_n| \leq |f|$, so by the dominated convergence theorem*,

$$\int_{E_n} f \, d\mu = \int_X f_n \, d\mu \to 0.$$

This is a contradiction.